TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 349, Number 3, March 1997, Pages 1081–1089
S 0002-9947(97)01902-8

SIGNED QUASI-MEASURES

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Abstract. Let X be a compact Hausdorff space and let $\mathcal{A}$ denote the subsets of X which are either open or closed. A quasi-linear functional is a map $\rho : C(X) \to \mathbf{R}$ which is linear on singly generated subalgebras and such that $|\rho(f)| \leq M\|f\|$ for some $M < \infty$. There is a one-to-one correspondence between the quasi-linear functional on $C(X)$ and the set functions $\mu : \mathcal{A} \to \mathbf{R}$ such that i) $\mu(\emptyset) = 0$, ii) If $A, B, A \cup B \in \mathcal{A}$ with A and B disjoint, then $\mu(A \cup B) = \mu(A) + \mu(B)$, iii) There is an $M < \infty$ such that whenever $\{U_\alpha\}$ are disjoint open sets, $\sum |\mu(U_\alpha)| \leq M$, and iv) if U is open and $\varepsilon > 0$, there is a compact $K \subseteq U$ such that whenever $V \subseteq U \setminus K$ is open, then $|\mu(V)| < \varepsilon$. The space of quasi-linear functionals is investigated and quasi-linear maps between two $C(X)$ spaces are studied.

Let X be a compact Hausdorff space and $C(X)$ the space of real-valued continuous functions on X. A map $\rho : C(X) \to \mathbf{R}$ is said to be a quasi-linear functional if ρ is linear on singly generated subalgebras and bounded in the sense that there exists an $M < \infty$ such that $|\rho(f)| \leq M\|f\|_u$ for all $f \in C(X)$. Let $\|\rho\|$ be the minimal such M. If ρ and η are quasi-linear functionals, we define $\rho + \eta$ by pointwise action on functions. In this fashion, the collection of all quasi-linear functionals becomes a normed linear space. Call this space $QL(X)$.

Notice that if ρ is quasi-linear, and $fg = 0$, then $\rho(f + g) = \rho(f) + \rho(g)$. In fact, if f and g are also positive, we have that the subalgebra generated by $f - g$ contains both f and g. In general, we can break f and g into positive and negative parts to get the result. Also notice that if c is a constant, $\rho(c + f) = \rho(c) + \rho(f)$. Thus, if f is constant on the support of g, we still have that $\rho(f+g) = \rho(f)+\rho(g)$.

Our goal is to find set functions that produce all quasi-linear functionals on $C(X)$. We will use an approach inspired by the techniques in [1] where the theory of *positive* quasi-linear functionals is presented. We use the notation $f \prec U$ when U is open to state that $0 \leq f \leq 1$ and f has support contained in U. We also use the notation sp f for the image of f.

Let $\mathcal{O}$ be the collection of open sets in X and $\mathcal{C}$ the collection of closed sets. Also, let $\mathcal{A} = \mathcal{O} \cup \mathcal{C}$. Thus $\mathcal{A}$ is the collection of subsets of X which are either open or closed.

Definition 1. A function $\mu : \mathcal{A} \to \mathbf{R}$ is called a signed quasi-measure if the following hold:

(i) $\mu(\emptyset) = 0$,
(ii) If $A, B \in \mathcal{A}$ are disjoint with $A \cup B \in \mathcal{A}$, then $\mu(A \cup B) = \mu(A) + \mu(B)$,

Received by the editors August 20, 1995.
1991 *Mathematics Subject Classification.* Primary 28C05.

(iii) There is a constant $M < \infty$ such that whenever $\{U_n\}$ is a finite disjoint collection of open sets, then $\sum |\mu(U_n)| \leq M$,

(iv) If an open set U and $\varepsilon > 0$ are given, there exists a closed set $K \subseteq U$ such that if V is an open set with $V \subseteq U \setminus K$, we have $|\mu(V)| < \varepsilon$.

We define $\|\mu\|$ to be the minimal M such that (iii) holds.

For future reference, we note that property (ii) above is equivalent to the following three statements.

a) If U and V are disjoint open sets, then $\mu(U \cup V) = \mu(U) + \mu(V)$.
b) If U and V are open with $X = U \cup V$, then $\mu(U) + \mu(V) = \mu(X) + \mu(U \cap V)$.
c) If U is open, then $\mu(X \setminus U) = \mu(X) - \mu(U)$.

This will allow us to define a quasi-measure by its action on only open sets.

Let $QM(X)$ denote the collection of all signed quasi-measures on X. If we define $\mu + \nu$ by action on sets, we see that $QM(X)$ is a normed linear space. We wish to show that in a natural way $QL(X)$ and $QM(X)$ are isomorphic as normed linear spaces, and are, in fact, Banach spaces.

Given a signed quasi-measure μ, we may define a new set function $|\mu|$ on open sets by

$$|\mu|(U) = \sup\{\sum |\mu(U_n)| : U_n \subseteq U \text{ are disjoint open sets}\}.$$

Then we see that $\|\mu\| = |\mu|(X)$. It is important to note here that $|\mu|$ need not yield a quasi-measure. In particular, it is impossible to define $|\mu|$ on closed sets so that (ii) holds. An example of this will be seen later. It is clear, however, that (i) and (iii) hold for $|\mu|$. We will see later that (iv) does also.

Proposition 2. *We have the following:*

a) If $\{A_\alpha\}_{\alpha \in A} \subseteq \mathcal{A}$ is a collection (possibly infinite) of disjoint subsets of $U \in \mathcal{O}$, then $\sum |\mu(A_\alpha)| \leq |\mu|(U)$.
b) If $U_1 \subseteq U_2 \subseteq \cdots$ are open, then $\mu\left(\bigcup_{i=1}^{\infty} U_i\right) = \lim_{i\to\infty} \mu(U_i)$.
c) If $\{U_\alpha\}$ is a collection (possibly infinite) of disjoint open sets, then $\mu(\bigcup U_\alpha) = \sum \mu(U_\alpha)$.

Proof. Both b) and c) are results of the regularity assumption (iv). For a), notice that there is a similar outer regularity for closed sets, so if $\{F_n\}_{n=1}^N$ are finitely many closed sets contained in U, and $\varepsilon > 0$, we may find disjoint open sets U_n with $F_n \subseteq U_n \subseteq U$ and $|\mu(U_n \setminus F_n)| < \varepsilon/N$. Then $\sum |\mu(F_n)| \leq \sum |\mu(U_n)| + \varepsilon \leq |\mu|(U) + \varepsilon$. For the general case, we may restrict to finitely many A_α, and approximate by closed sets using inner regularity for open sets. □

Now, let $f \in C(X)$ and $\alpha \in \mathbf{R}$, and define $\check{f}(\alpha) = \mu(f^{-1}(\alpha, +\infty))$ and $\hat{f}(\alpha) = \mu(f^{-1}[\alpha, +\infty))$. Notice that $\mu(f^{-1}(\alpha, \beta)) = \check{f}(\alpha) - \hat{f}(\beta)$ and $\mu(f^{-1}[\alpha, \beta]) = \hat{f}(\alpha) - \check{f}(\beta)$.

Proposition 3. *Let $f \in C(X)$. Then*

a) $\check{f}$ is continuous from the right and $\hat{f}$ is continuous from the left.
b) $\check{f}(\alpha^-) = \hat{f}(\alpha)$ and $\hat{f}(\alpha^+) = \check{f}(\alpha)$.
c) $\check{f}$ and $\hat{f}$ agree except at countably many $\alpha \in \mathbf{R}$.
d) If $\check{f}(\alpha) = \hat{f}(\alpha)$, then $\check{f}$ is continuous at α.
e) $\check{f}$ is of bounded variation with variation less than $\|\mu\|$.

Proof. a) If α_n decreases to α, then

$$\begin{aligned}\check{f}(\alpha) &= \mu\left(f^{-1}(\alpha,+\infty)\right) \\ &= \lim_{n\to\infty}\mu\left(f^{-1}(\alpha_n,+\infty)\right) \\ &= \lim_{n\to\infty}\check{f}(\alpha_n).\end{aligned}$$

Thus $\check{f}$ is continuous from the right. Since $\hat{f}(\alpha) = \mu(X) - (-f)^{\vee}(-\alpha)$, $\hat{f}$ is continuous from the left.

b) Let α_0 and $\varepsilon > 0$ be given and let $U = f^{-1}(\alpha_0, +\infty)$. Pick K as in iv) of the definition of quasi-measure. Let β be the minimum value of f on K. Then $\alpha_0 < \beta$. If $\alpha_0 < \alpha < \beta$, we have that $f^{-1}(\alpha_0, \alpha) \subseteq U \setminus K$, so $|\check{f}(\alpha_0) - \hat{f}(\alpha)| = |\mu\left(f^{-1}(\alpha_0,\alpha)\right)| < \varepsilon$. This shows that $\hat{f}(\alpha_0^+) = \check{f}(\alpha_0)$.

c) Notice that $\sum|\hat{f}(\alpha) - \check{f}(\alpha)| = \sum|\mu(f^{-1}(\{\alpha\}))| \le \|\mu\|$, by Proposition 2. Thus the set of α where $\hat{f}(\alpha) \neq \check{f}(\alpha)$ is at most countable.

d) This follows from parts a) and b).

e) If $\{(\alpha_n, \beta_n)\}$ is a disjoint collection of intervals, then

$$\begin{aligned}\sum|\check{f}(\alpha_n) - \check{f}(\beta_n)| &\le \sum|\check{f}(\alpha_n) - \hat{f}(\beta_n)| + \sum|\check{f}(\beta_n) - \hat{f}(\beta_n)| \\ &= \sum\left|\mu\left(f^{-1}(\alpha_n,\beta_n)\right)\right| + \sum\left|\mu\left(f^{-1}(\{\beta_n\})\right)\right| \\ &\le \|\mu\|\end{aligned}$$

by a) of Proposition 2. Thus $\check{f}$ is of bounded variation with variation at most $\|\mu\|$.

□

Since $\check{f}$ is of bounded variation, there is a signed measure μ_f on $\mathbf{R}$ such that $\mu_f(\alpha,\beta) = \check{f}(\alpha) - \check{f}(\beta^-) = \check{f}(\alpha) - \hat{f}(\beta) = \mu(f^{-1}(\alpha,\beta))$ and $\|\mu_f\| = |\mu_f|(\mathbf{R}) \le \|\mu\|$. If O is any open set in $\mathbf{R}$, we may write O as a disjoint union of open intervals to see that $\mu_f(O) = \mu\left(f^{-1}(O)\right)$. It follows that μ_f is concentrated on sp f.

If $f \in C(X)$ and $\varphi \in C(\text{sp}\, f)$, we let $\varphi^*\mu_f$ denote the image measure of μ_f under the map φ. The following lemma simplifies the proof of Proposition 3.2 of [1].

Lemma 4. *We have that* $\mu_{\varphi\circ f} = \varphi^*\mu_f$.

Proof. Let $O \subseteq \mathbf{R}$ be open. Then

$$\begin{aligned}\mu_{\varphi\circ f}(O) &= \mu\left((\varphi\circ f)^{-1}(O)\right) \\ &= \mu\left(f^{-1}\left(\varphi^{-1}(O)\right)\right) \\ &= \mu_f(\varphi^{-1}(O)) \\ &= (\varphi^*\mu_f)(O).\end{aligned}$$

□

Now we may define the functional $\rho_\mu(f) = \int_{\mathbf{R}} i \, d\mu_f$ where $i : \mathbf{R} \to \mathbf{R}$ is the function $i(x) = x$. Since μ_f is concentrated on sp f, we have $|\rho_\mu(f)| \le \int_{\mathbf{R}} |i| \, d|\mu_f| \le$

$\|\mu\|\|f\|_u$, so ρ_μ is bounded with $\|\rho_\mu\| \leq \|\mu\|$. Also

$$\begin{aligned}\rho_\mu(\varphi\circ f) &= \int_{\mathbf{R}} i\ d\mu_{\varphi\circ f}\\ &= \int_{\mathbf{R}} i\ d\varphi^*\mu_f\\ &= \int_{\mathbf{R}} \varphi\ d\mu_f,\end{aligned}$$

so $\rho_\mu(\varphi\circ f+\psi\circ f) = \int_{\mathbf{R}}\varphi+\psi\ d\mu_f = \rho_\mu(\varphi\circ f)+\rho_\mu(\psi\circ f)$. Thus ρ_μ is a quasi-linear functional on $C(X)$.

Theorem 5. *The map $\mu \to \rho_\mu$ is an isometric isomorphism of the normed linear space $QM(X)$ onto $QL(X)$.*

Proof. It is easy to see that this map is linear. We show that it is onto $QL(X)$.

Suppose ρ is a quasi-linear functional on $C(X)$.

Claim 1: *If U is open in X and $\varepsilon > 0$, there is a closed $K \subseteq U$ such that if $f \in C(X)$, $\|f\|_u \leq 1$,* supp $f \subseteq U$, *and $f = 0$ on K, then $|\rho(f)| < \varepsilon$.*

Suppose no such K exists for some $\varepsilon > 0$. Pick $f_1 \in C(X)$ such that $\|f_1\|_u \leq 1$, supp $f_1 \subseteq U$, and $|\rho(f_1)| \geq \varepsilon$. Pick $|a_1| = 1$ such that $\rho(a_1f_1) = |\rho(f_1)| \geq \varepsilon$. Now $K_1 =$ supp f_1 fails the conditions of the claim, so there is an $f_2 \in C(X)$ supported in U such that $\|f_2\| \leq 1$, $f_2 = 0$ on K_1, and $|\rho(f_2)| \geq \varepsilon$. Pick $|a_2| = 1$ such that $\rho(a_2f_2) = |\rho(f_2)| \geq \varepsilon$. Since $f_2 = 0$ on the support of f_1, we have that $\rho(a_1f_1 + a_2f_2) = \rho(a_1f_1) + \rho(a_2f_2) \geq 2\varepsilon$ and $\|a_1f_1 + a_2f_2\|_u \leq 1$. Continuing by induction, we may find $f_n \in C(X)$ supported in U that vanishes on the support of $a_1f_1 + a_2f_2 + \cdots + a_{n-1}f_{n-1}$ and $\|f_n\| \leq 1$, while $\rho(a_nf_n) = |\rho(f_n)| \geq \varepsilon$. But then $\rho(\sum_{i=1}^{n} a_if_i) \geq n\varepsilon$, which violates the boundedness of ρ for large n.

Claim 2: *For U open, $\lim_{f\prec U}\rho(f)$ exists where the f are ordered pointwise.*

We show that this net is a Cauchy net. In fact, let $\varepsilon > 0$ and let $K \subseteq U$ be the closed set of Claim 1. Let f be any function such that $f = 1$ on K and $f \prec U$. If $f \leq g, h \prec U$, then pick k with $k = 1$ on supp $g\,\cup$ supp h, and $k \prec U$. Then $g - k$ and $h-k$ vanish on K, so we have that $|\rho(g)-\rho(h)| \leq |\rho(g)-\rho(k)|+|\rho(h)-\rho(k)| = |\rho(g - k)| + |\rho(h - k)| \leq 2\varepsilon$.

Define $\mu(U) = \lim_{f\prec U}\rho(f)$ for U open in X.

Claim 3: *μ is a signed quasi-measure on X.*

Easily, $\mu(U\cup V) = \mu(U) + \mu(V)$ if U and V are disjoint. Also $\mu(\emptyset) = 0$. Notice also that $\mu(X) = \rho(1)$. We next show property b) after the definition of a signed quasi-measure.

Suppose that $U\cup V = X$. Pick $C \subseteq U$ and $K \subseteq V$, closed such that $C\cup K = X$. Pick $f_0 \prec U$, $g_0 \prec V$ with $f_0 = 1$ on C, $g_0 = 1$ on K, and such that $f_0 \leq f \prec U$ implies $|\rho(f) - \mu(U)| < \varepsilon$ and $g_0 \leq g \prec V$ implies $|\rho(g) - \mu(V)| < \epsilon$. Let $h_0 \prec U\cap V$ with $h_0^2 = 1$ on $C\cap K$ and such that $h_0^2 \leq h \prec U\cap V$ implies that $|\rho(h) - \mu(U\cap V)| < \varepsilon$. Now, set $f = \max\{f_0, h_0\}$ and $g = \max\{g_0, h_0\}$. Then $f \prec U$, $g \prec V$, and $fg \prec U\cap V$, so $|\rho(f) - \mu(U)| < \varepsilon$, $|\rho(g) - \mu(V)| < \varepsilon$ and $|\rho(fg) - \mu(U\cap V)| < \varepsilon$. Also, since $f = 1$ on C and $g = 1$ on K, and $C\cup K = X$, we have that $(1 - f)(1 - g) = 0$, so $\rho((1 - f) + (1 - g)) = \rho(1 - f) + \rho(1 - g)$ and

$\rho(f+g) = \rho(1+fg)$. This gives that $\rho(f)+\rho(g) = \rho(f+g) = \rho(1)+\rho(fg)$, which shows that $|\mu(U)+\mu(V)-\mu(X)-\mu(U\cap V)| < 3\varepsilon$.

Now suppose that $\{U_n\}_{n=1}^N$ is a finite, disjoint collection of open sets. Let $\varepsilon > 0$ be given and choose $f_n \prec U_n$ such that $|\rho(f_n)-\mu(U_n)| < \varepsilon$. Now choose $|a_n| = 1$ such that $|\rho(f_n)| = \rho(a_n f_n)$. Then $|\rho\left(\sum a_n f_n\right)| \leq \|\rho\|$, so $\sum |\mu(U_n)| \leq \|\rho\| + N\varepsilon$. Now let $\varepsilon \to 0$.

Finally, if U an open set and $\varepsilon > 0$ are given, choose $K \subseteq U$ as in Claim 1, and argue as in the previous paragraph to show that if $\{U_n\}$ are disjoint and open with $U_n \subseteq U \setminus K$, then $\sum |\mu(U_n)| < \varepsilon$. In particular, if $V \subseteq U \setminus K$ is open, $|\mu(V)| < \varepsilon$.

Notice that $\|\mu\| \leq \|\rho\|$.

Claim 4: *We have that* $\rho = \rho_\mu$.

For each $f \in C(X)$, the map $\varphi \to \rho(\varphi \circ f)$ is bounded and linear on $C(\mathrm{sp}\ f)$. Thus, there is a signed measure ν_f on sp f such that

$$\rho(\varphi \circ f) = \int_{\mathbf{R}} \varphi \, d\nu_f$$

for all $\varphi \in C(\mathrm{sp}\ f)$. Since $\rho(f) = \int_{\mathbf{R}} i \, d\nu_f$, we need only show that $\nu_f = \mu_f$ for each $f \in C(X)$. Notice that both measures are measures on $\mathbf{R}$.

Suppose that O is an open set in $\mathbf{R}$. Pick closed sets $C_n \subseteq O$ such that $C_n \subseteq \mathrm{int}(C_{n+1})$ and $O = \bigcup C_n$. Choose $\varphi_n \prec O$ such that $\varphi_n = 1$ on C_n. Since X is compact, the sequence $\varphi_n \circ f$ is cofinal in the collection of functions g such that $g \prec f^{-1}(O)$. Thus

$$\begin{aligned} \nu_f(O) &= \lim_{n\to\infty} \int \varphi_n \, d\nu_f \\ &= \lim_{n\to\infty} \rho(\varphi_n \circ f) \\ &= \mu\left(f^{-1}(O)\right) \\ &= \mu_f(O) \end{aligned}$$

giving the required equality of measures. Thus $\|\mu\| \leq \|\rho\| = \|\rho_\mu\| \leq \|\mu\|$.

This shows that the map $\mu \to \rho_\mu$ is onto $QL(X)$, and in fact, that any $\rho \in QL(X)$ is the image of some $\mu \in QM(X)$ of the same norm. If we show that our map is one-to-one, we will be finished.

Assume $\rho_\mu = 0$. Then $\mu_f = 0$ for all $f \in C(X)$. Thus $\check{f}(\alpha) = \mu_f(\alpha, +\infty) = 0$ for all $\alpha \in \mathbf{R}$. If, now, $U \subseteq X$ is open and $\varepsilon > 0$, pick $K \subseteq U$ as in part (iv) of the definition of a signed quasi-measure. Choose any $f \in C(X)$ with $K \prec f \prec U$. Then

$$\begin{aligned} |\mu(U)| &\leq |\mu(U)-\mu(K)| + |\check{f}(\tfrac{1}{2}) - \mu(K)| + |\check{f}(\tfrac{1}{2})| \\ &= |\mu(U\setminus K)| + |\mu\left((f^{-1}(\tfrac{1}{2}, +\infty)) \setminus K\right)| + |\check{f}(\tfrac{1}{2})| \\ &\leq 2\varepsilon. \end{aligned}$$

Thus $\mu(U) = 0$ for all open sets, so $\mu = 0$. □

It should be noted that in Claim 2 we actually showed a stronger form of regularity. If U is open and $\varepsilon > 0$, then there is a closed set $K \subseteq U$ with $|\mu|(U\setminus K) < \varepsilon$. Thus $|\mu|$ obeys part (iv) of the definition of a quasi-measure.

There is yet another representation of $QL(X)$ that is sometimes useful. For each $f \in C(X)$, let $M(\text{sp } f)$ denote the collection of regular Borel measures on the compact set sp f with the usual measure norm. Define $PM(X)$ to be

$$\Big\{(\nu_f) \in \prod_{f \in C(X)} M(\text{sp } f) : \nu_{\varphi\circ f} = \varphi^* \nu_f \text{ for } \varphi \in C(\text{sp } f) \text{ and } \sup \|\nu_f\| < \infty\Big\}.$$

Define a norm on $PM(X)$ by $\|(\nu_f)\| = \sup \|\nu_f\|$. Then it is easy to see that $PM(X)$ is a Banach space since this is true for each $M(\text{sp } f)$.

If μ is a signed quasi-measure, then the collection (μ_f) in the definition of ρ_μ is an element of $PM(X)$ with $\|(\mu_f)\| \le \|\mu\|$. The induced map from $QM(X)$ to $PM(X)$ is evidently linear.

On the other hand, if $(\nu_f) \in PM(X)$, we may define $\rho(f) = \int_{\mathbf{R}} i \, d\nu_f$. Then the argument just before the statement of the theorem shows that $\rho \in QL(X)$ with $\|\rho\| \le \|(\nu_f)\|$. This, with the last paragraph shows that $PM(X)$ is isometrically isomorphic to both $QM(X)$ and $QL(X)$.

In particular $QM(X)$ is a Banach space. We can make it an ordered Banach space by taking the positive cone to be the collection of positive quasi-measures. The norm on this space is additive on the positive cone, but $QM(X)$ does not have to be a lattice. Thus, $QM(X)$ need not be an L-space. For example, in [3], Aarnes finds positive $\{0,1\}$-valued quasi-measures $\mu_1, \mu_2, \mu_3, \mu_4$ with $\mu_1 + \mu_3 = \mu_2 + \mu_4$. Since $\{0,1\}$-valued quasi-measures are extremal, there will then be no supremum of $\{\mu_1, \mu_2\}$.

For future convenience, we define the notation $\langle \mu, f\rangle = \langle \rho_\mu, f\rangle = \rho_\mu(f)$.

Another aspect of the failure of the lattice property is that the set function $|\mu|$ need not have an extension to a positive quasi-measure on X. An example of this is given next.

Example. Let $X = [0,1] \times [0,1]$. In [2] and [6], it is shown how to construct the so-called three point quasi-measures. This is done as follows. A subset A of X is said to be solid if both A and $X \setminus A$ are connected. If $C = \{x_1, x_2, x_3\}$ is a set with three elements, we define μ_C on solid sets by

$$\mu_C(A) = \begin{cases} 0 & \text{if } \operatorname{card}(A \cap C) \le 1, \\ 1 & \text{if } \operatorname{card}(A \cap C) \ge 2. \end{cases}$$

There is then a unique extension of μ_C to a $\{0,1\}$-valued quasi-measure on X.

Now let $x_1 = (0,0), x_2 = (1,0), x_3 = (1,1)$, and $x_4 = (0,1)$. Let $C = \{x_1, x_2, x_3\}$, $D = \{x_2, x_3, x_4\}$, and $\mu = \mu_C - \mu_D$. If we let $U_1 = [0,1] \times [0, \frac{1}{2})$ and $U_2 = [0,1] \times (\frac{1}{2}, 1]$, we see that $2 = |\mu(U_1)| + |\mu(U_2)| \le |\mu|(X) = \|\mu\| \le 2$. Hence, $|\mu|(X) = 2$. We show that $|\mu|$ cannot be extended to closed sets in such a way that it is a positive quasi-measure. In particular, (ii) does not hold in the definition of a quasi-measure.

Assume such an extension exists. Let $V_1 = [0, \frac{3}{4}) \times [0,1]$ and $V_2 = (\frac{1}{4}, 1] \times [0,1]$. Write $K_1 = X \setminus V_1$ and $K_2 = X \setminus V_2$. Since $\mu_C(V_1) = \mu_D(V_1) = 0$, we see that $|\mu|(V_1) = 0 = |\mu|(V_1 \cap V_2)$. Hence $|\mu|(K_1) = |\mu|(K_1 \cup K_2) = 2$. Since K_1 and K_2 are disjoint, we must have that $|\mu|(K_2) = 0$, in other words that $|\mu|(V_2) = 2$. However this is not the case. In fact, $|\mu|(V_2) = 0$.

To see this we show that if $U \subseteq V_2$ is open, then $\mu(U) = 0$; that is $\mu_C(U) = \mu_D(U)$. Using symmetry and the fact that both μ_C and μ_D take on only 0 and

1 as values, we only show that $\mu_C(U) = 0$ implies $\mu_D(U) = 0$. Furthermore, by considering components, it is enough to show this for connected open sets.

Suppose, then, that U is connected and let K be any connected closed subset of U. Let $\hat{K}$ be the solid hull of K as a subset of V_2 as in [2]. Then $\hat{K}$ is a solid closed set in V_2 containing K. If $\mu_C(\hat{K}) = 0$, then $\{x_2, x_3\} \cap \hat{K}$ has at most one element, so $\mu_D(K) \leq \mu_D(\hat{K}) = 0$. On the other hand, if $\mu_C(\hat{K}) = 1$, there is some interior component W of $X \setminus K$ with $\mu_C(W) = 1$. This follows since $\mu_C(K) = 0$. Since $W \subseteq \hat{K} \subseteq V_2$, we have that $x_2, x_3 \in W$. Since W is solid, we then have that $\mu_D(W) = 1$, so $\mu_D(K) \leq \mu_D(X \setminus W) = 0$. Finally, $\mu_D(U)$ is the supremum of $\mu_D(K)$ as above, so $\mu_D(U) = 0$. □

It would be nice to know that every signed quasi-measure is the difference of two positive quasi-measures. However, the failure of the lattice property and the fact that $|\mu|$ need not be a quasi-measure brings this into question. At this point the issue remains open.

We now turn to another topology on $QM(X)$ that is very useful.

Definition 6. The weak-* topology on $QM(X)$ is the weakest topology making each map $\mu \to \langle \mu, f \rangle$ continuous where f ranges over $C(X)$.

Since each such map is linear and the collection of these maps separates points of $QM(X)$, we obtain a locally convex topology on $QM(X)$ where a net μ_α converges to μ if and only if $\langle \mu_\alpha, f \rangle$ converges to $\langle \mu, f \rangle$ for every $f \in C(X)$. This topology has been studied on the space of positive quasi-measures in [3].

Proposition 7. *Let μ_α be a net in $QM(X)$ and $\mu \in QM(X)$. Let $(\mu_{\alpha,f})$ and (μ_f) be the corresponding elements of $PM(X)$. Then μ_α converges to μ in the weak-* topology if and only if $\mu_{\alpha,f}$ converges to μ_f in the weak-* topology of $M(\mathrm{sp}\ f)$ for each $f \in C(X)$. Thus, the unit ball in $QM(X)$ is weak-* compact.*

Proof. If $\mu_{\alpha,f}$ converges to μ_f for each $f \in C(X)$, then $\langle \mu_\alpha, f \rangle = \int_{\mathbf{R}} i \ d\mu_{\alpha,f}$ converges to $\int_{\mathbf{R}} i \ d\mu_f = \langle \mu, f \rangle$.

Conversely, if μ_α converges to μ weak-*, then for each $\varphi \in C(\mathrm{sp}\ f)$, we have

$$\begin{aligned} \lim_\alpha \int_{\mathbf{R}} \varphi \ d\mu_{\alpha,f} &= \lim_\alpha \langle \mu_\alpha, \varphi \circ f \rangle \\ &= \langle \mu, \varphi \circ f \rangle \\ &= \int_{\mathbf{R}} \varphi \ d\mu_f. \end{aligned}$$

Thus $\mu_{\alpha,f}$ converges to μ_f in the weak-* topology.

Since the map $\mu_f \to \varphi^* \mu_f$ is weak-* continuous, the compactness of the unit ball of $QM(X)$ follows from the compactness of the unit balls of $M(\mathrm{sp}\ f)$. □

Definition 8. Let X and Y be compact Hausdorff spaces. A quasi-linear map from $C(X)$ to $C(Y)$ is a map, T, which is linear on each singly generated subalgebra of $C(X)$ and which is bounded in the sense that there is an $M < \infty$ with $\|T(f)\| \leq M\|f\|$. If, in addition, T is multiplicative on each singly generated subalgebra, we say that T is a quasi-homomorphism.

For example, let ρ be a positive quasi-linear functional on X and let Y be any compact Hausdorff space. Define $T_\rho : C(X \times Y) \to C(Y)$ by $T_\rho(f)(y) = \rho(f^y)$ where $f^y(x) = f(x, y)$. It is noted in [5] that T_ρ is a quasi-linear map. If $\eta : C(Y) \to \mathbf{R}$

is an additional quasi-linear functional, it is also noted there that the composition of T_ρ and η need not be quasi-linear.

Proposition 9. *There is a one to one correspondence between quasi-linear maps from $C(X)$ to $C(Y)$ and norm-bounded functions $Y \to QM(X)$ which are weak-* continuous. Specifically, if $y \to \mu_y$ is a bounded, weak-* continuous map of Y into $QM(X)$, the corresponding quasi-linear map is $T(f)(y) = \langle \mu_y, f \rangle$.*

The proof of this is evident. For comparison, there is a similar correspondence between the quasi-homomorphisms from $C(X)$ to $C(Y)$ and weak-* continuous functions $Y \to X^*$ where X^* is the collection of $\{0,1\}$-quasi-measures on X. See [4] for details.

Proposition 10. *Let $T : C(X) \to C(Y)$ be a quasi-homomorphism and $S : C(Y) \to C(Z)$ be a quasi-linear map. Then the composition $S \circ T : C(X) \to C(Z)$ is a quasi-linear map. If $y \to \mu_y$ is the map from Y to X^* corresponding to T, $z \to \nu_z$ the map from Z into $QM(Y)$ corresponding to S, and $z \to \omega_z$ the map from Z to $QM(X)$ corresponding to $S \circ T$, then for U open in X,*

$$\omega_z(U) = \nu_z(\{y \in Y : \mu_y(U) = 1\}).$$

Proof. That the composition is quasi-linear is evident.

Let $U \subseteq X$ be open and $\varepsilon > 0$. Let $W = \{y : \mu_y(U) = 1\}$.

Claim: *For each $y \in W$ there is a compact $K_y \subseteq U$ and open set $y \in V_y$ such that $y' \in V_y$ implies $\mu_{y'}(K_y) = 1$.*

Since $\mu_y(U) = 1$, there is a compact $K_1 \subseteq U$ with $\mu_y(K_1) = 1$. Let $f \in C(X)$ with $K_1 \prec f \prec U$. Then $\langle \mu_y, f \rangle = 1$, so by weak-* continuity, there is an open set $y \in V$ such that $y' \in V$ implies that $\langle \mu_{y'}, f \rangle > \frac{2}{3}$. Let $K_y = \{x : f(x) \geq \frac{1}{3}\}$. Since each $\mu_{y'}$ is a $\{0,1\}$-quasi-measure, $\mu_{y'}(K_y) = 1$ for $y' \in V$.

In particular, we see that W is open in Y. This also shows that if $A \subseteq X$ is closed then $\{y : \mu_y(A) = 1\}$ is closed in Y. Now we can find compact sets $K \subseteq U$ and $C \subseteq W$ such that $K \prec f \prec U$ implies $|\langle \omega_z, f \rangle - \omega_z(U)| < \varepsilon$ and $C \prec g \prec W$ implies $|\langle \nu_z, g \rangle - \nu_z(W)| < \varepsilon$.

For each $y \in C$, pick V_y and K_y as in the claim. Choose finitely many V_y to cover C, say $V_1, \cdots, V_n$. Let $K_1, \cdots, K_n$ be the corresponding K_y.

Set $L = K \cup K_1 \cup \cdots \cup K_n$ and find $L \prec f \prec U$. Recall $T(f)(y) = \langle \mu_y, f \rangle$ for $y \in Y$. If $y \in C$, say $y \in V_j$, $1 \leq \mu_y(K_j) \leq \mu_y(L) \leq \langle \mu_y, f \rangle \leq \mu_y(U) \leq 1$, so $C \prec T(f)$. Also, if F is the support of f, then $E = \{y : \mu_y(F) = 1\}$ is closed and contained in W. If $y \notin E$, $\mu_y(F) = 0$, so $\langle \mu_y, f \rangle = 0$. Thus, the support of $T(f)$ is contained in $E \subseteq W$. Hence $C \prec T(f) \prec W$.

Thus, $|\omega_z(U) - \langle \omega_z, f \rangle| < \varepsilon$ and $|\nu_z(W) - \langle \nu_z, T(f) \rangle| < \varepsilon$. However, $\langle \omega_z, f \rangle = S \circ T(f)(z) = \langle \nu_z, T(f) \rangle$. Hence $|\omega_z(U) - \nu_z(W)| < 2\varepsilon$. Now let ε go to 0. □

Another interpretation of this result may be obtained by noting that a quasi-homomorphism $T : C(X) \to C(Y)$ induces an image transformation $q : \mathcal{A}(X) \to \mathcal{A}(Y)$ (see [4] for details). In particular, for $A \in \mathcal{A}(X)$, we have $q(A) = \{y : \mu_y(A) = 1\}$. For $\nu \in QM(Y)$, we can then define $q^*\nu \in QM(X)$ by $q^*\nu(A) = \nu(q(A))$. The above proposition then states that $\omega_z = q^*\nu_z$ for $z \in Z$.

I would like to thank Johan Aarnes and Robert Wheeler for their encouragement to pursue this paper.

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